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STABILITY OF A VISCOELASTIC ROD SUBJECT TO A RANDOM STATIONARY LONGITUDINAL FORCE†

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The stability of a rod made of a non-aging viscoelastic material whose relaxation kernel can be expressed as the sum of exponents is investigated. The exact conditions for the stability of motion with respect to the statistical moments of the deflection amplitude of the rod, subject to a compressing force whose random component is proportional to white noise, are obtained.

THE STABILITY of viscoelastic rods is considered in [1], where the mean square dispersion of the external load and the measure of the creep of the material are both assumed to be small. The asymptotic method [2] was used to solve the problem under such assumptions, leading to results whose degree of accuracy has remained unknown. In [3, 4] a number of model problems concerned with the stability of motion of a rod have been considered. In [5] the second Lyapunov method was employed to analyse the stability of a rod made from a standard viscoelastic material subject to a longitudinal force of the type of white noise. The condition under which the rod is almost surely stable was found.

1. THE STABILITY OF A VISCOELASTIC ROD

The motion of a viscoelastic rod subject to a longitudinal force F can be described by the equation

$$EI\,(1-\Gamma)W'''' + (F_0+F_1)(W+W_0)'' + mW^{\cdot\cdot} + kW^{\cdot} = 0 \qquad (1.1)$$

$$\Gamma W = \int_0^t \Gamma(t-\tau)\,W(\tau)\,d\tau$$

Here k is the damping factor, which takes into account the external resistance to the motion of the rod, W_0 is the initial deflection of the axis of the rod, and F_0 and $F_1(t)$ are the deterministic component of the compressive force (which is constant in time) and a random pulsation with zero expectation value. The remaining notation is the generally accepted one.

We assume that the rod is hinged at each end and has the following initial deflection W_0, as well as the additional deflection W of its axis at the initial instant of time:

$$W_0(x) = f_0 \sin\frac{\pi}{l}x, \quad W(0,x) = f(0)\sin\frac{\pi}{l}x$$

We shall seek a solution of Eq. (1.1) in the form of the same sinusoid, whose amplitude can be determined from the equation

$$f^{\cdot\cdot} + 2\varepsilon f^{\cdot} + \omega^2\,[(1-\Gamma)f - (\alpha+\alpha_1)(f+f_0)] = 0 \qquad (1.2)$$

$$\varepsilon = \frac{k}{2m}, \quad \omega^2 = \frac{\pi^4 EI}{ml^4}, \quad \alpha = \frac{F_0 l^2}{\pi^2 EI}, \quad \alpha_1(t) = \frac{F_1(t)\,l^2}{\pi^2 EI}$$

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From now on we shall assume that the equality

$$\Gamma f = \sum_{i=1}^{n} \varkappa_i L_i \int_0^t e^{-\varkappa_i(t-\tau)} f(\tau)\, d\tau$$

is satisfied, $\varkappa_i$ and L_i constants, which characterize the viscous properties of the material.

Using the substitution

$$z_i^{\cdot} = \varkappa_i L_i \int_0^t e^{-\varkappa_i(t-\tau)} f(\tau)\, d\tau$$

we can represent Eq. (1.2) as the following system of first-order differential equations:

$$\begin{aligned} f_1^{\cdot} &= f_2 \\ f_2^{\cdot} &= -2\varepsilon f_2 - \omega^2 \Big[f_1 - \sum_{i=1}^{n} z_i - (\alpha + \alpha_1)(f_1 + f_0)\Big] \\ z_i^{\cdot} &= \varkappa_i (L_i f_1 - z_i), \quad i = 1, 2, \ldots, n. \end{aligned} \tag{1.3}$$

We assume that the random pulsation of the longitudinal force is proportional to Gaussian white noise $\xi(t)$, i.e. $\alpha_1(t) = \beta\xi(t)$, where β is a deterministic constant. Then the system of equations (1.3) describes the evolution of an $(n+2)$-dimensional Markov process.

For this process, we write down the Fokker–Planck–Kolmogorov equation

$$\begin{aligned} \frac{\partial p}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} [\varkappa_i (z_i - L_i f_1) p] - \frac{\partial}{\partial f_1}(f_2 p) + \frac{\partial}{\partial f_2}\Big\{\Big[2\varepsilon f_2 + \omega^2\Big(f_1 - \sum_{i=1}^{n} z_i - \\ - \alpha\Big)(f_1 + f_0)\Big] p\Big\} + \frac{\beta^2\omega^4}{2}\frac{\partial^2}{\partial f_2^2}[(f_1 + f_0)^2 p] \end{aligned} \tag{1.4}$$

Using this equation, we can write down the equations for the statistical moments of f_1, f_2, z_i $(i = 1, 2, \ldots, n)$ of arbitrary order. In particular, for the first- and second-order moments, we obtain the following systems of equations:

$$\begin{aligned} \frac{d\langle f_1\rangle}{dt} &= \langle f_2\rangle \\ \frac{d\langle f_2\rangle}{dt} &= -2\varepsilon\langle f_2\rangle - \omega^2\Big[\langle f_1\rangle - \sum_{i=1}^{n}\langle z_i\rangle - \alpha(\langle f_1\rangle + f_0)\Big] \\ \frac{d\langle z_i\rangle}{dt} &= \varkappa_i (L_i\langle f_1\rangle - \langle z_i\rangle), \quad i = 1, 2, \ldots, n \end{aligned} \tag{1.5}$$

$$\begin{aligned} \frac{d}{dt}\langle f_1^2\rangle &= 2\langle f_1 f_2\rangle \\ \frac{d}{dt}\langle f_1 f_2\rangle &= \langle f_2^2\rangle - \Big\{2\varepsilon\langle f_1 f_2\rangle + \omega^2\Big[\langle f_1^2\rangle - \sum_{i=1}^{n}\langle f_1 z_i\rangle - \alpha(\langle f_1^2\rangle + \langle f_1\rangle f_0)\Big]\Big\} \\ \frac{d}{dt}\langle f_2^2\rangle &= -2\Big\{2\varepsilon\langle f_2^2\rangle + \omega^2\Big[\langle f_1 f_2\rangle - \sum_{i=1}^{n}\langle f_2 z_i\rangle - \alpha(\langle f_1 f_2\rangle + \langle f_2\rangle f_0)\Big]\Big\} + \\ &\quad + \beta^2\omega^4[\langle f_1^2\rangle + 2\langle f_1\rangle f_0 + f_0^2] \\ \frac{d}{dt}\langle z_k z_j\rangle &= -[\varkappa_j(\langle z_k z_j\rangle - L_j\langle z_k f_1\rangle) + \varkappa_k(\langle z_k z_j\rangle - L_k\langle z_j f_1\rangle)] \\ \frac{d}{dt}\langle z_k f_1\rangle &= -[\varkappa_k(\langle z_k f_1\rangle - L_k\langle f_1^2\rangle)] + \langle f_2 z_k\rangle \\ \frac{d}{dt}\langle z_k f_2\rangle &= -[\varkappa_k(\langle z_k f_2\rangle - L_k\langle f_2 f_1\rangle)] - \\ &\quad - \Big\{2\varepsilon\langle z_k f_2\rangle + \omega^2\Big[\langle z_k f_1\rangle - \sum_{i=1}^{n}\langle z_k z_i\rangle - \alpha(\langle z_k f_1\rangle + \langle z_k\rangle f_0)\Big]\Big\} \end{aligned} \tag{1.6}$$

Here and henceforth the average over the set of samples is denoted by angle brackets.

The Fokker–Planck–Kolmogorov equation enables us not only to obtain the equations for the statistical moments of the deflection amplitude of the rod, but also, in particular, to solve the problem of the excursions of $f(t)$ beyond the boundaries of the domain of admissible values, which enables one to answer to question of the stability of the rod over a finite time interval. If the problem can be reduced solely to obtaining the equations in terms of the moments, then, in order to do so, one can use the equations of motion (1.3) directly (see [6]).

Consider a rod with zero initial deflection of the axis ($f_0 = 0$). The stability of the straight-line state of equilibrium of the rod under perturbations of the initial conditions with respect to the first- and second-order statistical moments is determined by the signs of the real parts of the characteristic roots of the equations

$$| \mathbf{A}^{(m)} - \lambda \mathbf{E} | = 0$$

where $\mathbf{A}^{(m)}$ denotes the matrix formed by the constant coefficients of the system of differential equations for the moments of the corresponding order and $\mathbf{E}$ is the identity matrix.

We take the relaxation kernel of the material of the rod of the form

$$\Gamma (t - \tau) = \varkappa L e^{-\varkappa(t-\tau)}$$

Then, considering the system of equations with respect to the expectation values $\langle f_1 \rangle$, $\langle f_2 \rangle$, $\langle z_1 \rangle$, one can verify that the rod is stable, the stability being asymptotic with respect to the second-order moments [8], if the condition

$$\alpha < 1 - L$$

is satisfied.

We remark that this relation is the same as the condition of stability for a viscoelastic rod in the deterministic formulation of the problem.

Next, considering the system of equations with respect to the statistical moments of order two, we obtain the condition (for $\delta > 0$)

$$-\beta^2 \omega^4 \eta < \delta 2 \, [2\varepsilon\omega^2 (1 - \alpha) + \varkappa\omega^2 L + 2\varepsilon\varkappa (\varkappa + 2\varepsilon)] \tag{1.7}$$

under which the rod turns out to be mean-square asymptotically stable.

Here

$$\eta = 1 - L - \alpha + \varkappa (\varkappa + 2\varepsilon)/\omega^2, \ \delta = 1 - L - \alpha$$

2. SPECIAL CASES

Case 1. An elastic rod. For $\varkappa = L = 0$, inequality (1.7) takes the form

$$\beta^2 < 4\varepsilon/\omega^2 (1 - \alpha) \tag{2.1}$$

which is identical with the result obtained in [7].

Hence it follows that the random dispersion of the longitudinal force increases as the external deformation increases and the frequency ω of characteristic oscillations decreases.

Case 2. A viscoelastic rod without external deformation. Setting $\delta > 0$ for $\varepsilon = 0$, we can find from (1.7) that

$$\varkappa\beta^2 < 2 \, (\varkappa/\omega)^2 L\delta/\eta_0 \tag{2.2}$$

or

$$\alpha < 1 - L - (\varkappa/\omega)^2 \Lambda^{-1}, \ \Lambda = 2 \, (\varkappa/\omega)^2 L/(\varkappa\beta^2) - 1 \tag{2.3}$$

where

$$\eta_0 = 1 - L - \alpha + (\varkappa/\omega)^2, \quad \Lambda > 0$$

Inequalities (2.2) and (2.3) indicate that the values of β^2 or α that ensure the mean-square stability of the rod depend not only on L, which determines the limiting stress relaxation in the viscoelastic rod, but also on the ratio $\varkappa/\omega$. In particular, if

$$(\varkappa/\omega)^2\Lambda^{-1} = 1 - L$$

then the rod can be stable only for a constant stretching component of the longitudinal force.

If we employ the asymptotic method [1, 3] to determine the stability conditions similar to (2.2) and (2.3), we get (for $1-\alpha>0$)

$$\varkappa\beta^2 < 2\,(\varkappa/\omega)^2 L\,(1-\alpha)(1-\alpha+\varkappa^2/\omega^2)^{-1}$$
$$\alpha < 1 - (\varkappa/\omega)^2\Lambda^{-1}$$

Comparing these inequalities with (2.2) and (2.3), one can note that they yield similar results for small L.

Therefore, when applying the asymptotic method, it suffices to stipulate that L should be small, without imposing any restrictions on the parameter $\varkappa$, which characterizes the relaxation time of the material.

To compare the influence of the resistance of the motion, which is proportional to the rate of variation of the deflection, and the viscous properties of the material upon the stability of the motion of the rod, we represent (1.7) in the form

$$\beta^2\omega < 2\,(1-L-\alpha)\{2\varepsilon/\omega + L\,(2\varepsilon/\omega + \varkappa/\omega)[1 - L - \alpha + \varkappa\,(\varkappa + 2\varepsilon)/\omega^2]^{-1}\} \tag{2.4}$$

We assume that ε/ω and $\varkappa/\omega$ are approximately equal to one another. Then

$$\beta^2\omega < 4\,(1-L-\alpha)(\varepsilon/\omega)[1 + {}^3/_2 L\,(1-L-\alpha+3\varepsilon^2/\omega^2)^{-1}]$$

If we assume that $\varepsilon^2/\omega^2 \approx 0$, then

$$\beta^2\omega < 4\,(\varepsilon/\omega)(1 + {}^1/_2 L - \alpha) \tag{2.5}$$

The constant L can take values between 0 and 1 inclusive ($L=0$ for an elastic medium and $L=1$ for a viscoelastic medium described by Maxwell's model).

The comparison of (2.5) with the similar inequality (2.1) for an elastic rod shows that the upper limit for $\beta^2\omega$ is much larger for a viscoelastic rod than for an elastic one.

But if we assume that the relations $\varkappa/\omega \ll \varepsilon/\omega$ and $\varepsilon\varkappa/\omega^2 \approx 0$ are satisfied, then from (2.4) we can obtain the condition of stability, which is exactly the same as that for an elastic rod.

Finally, we consider the case when $L=1$. Then inequality (1.7) takes the form

$$\beta^2\omega < -2\alpha\,\{2\varepsilon/\omega + (2\varepsilon/\omega + \varkappa/\omega)[\varkappa\,(\varkappa+2\varepsilon)/\omega^2 - \alpha]^{-1}\}$$

with

$$\varkappa\,(\varkappa + 2\varepsilon)/\omega^2 - \alpha > 0$$

Hence one can see that the rod can be stable when the expectation value of the longitudinal force is a constant (in time) tensile force.

Case 3. A weightless rod in a viscous medium. As $m\to 0$, the frequency ω of the characteristic oscillations increases without limit and the ratio $2\varepsilon/\omega^2$ remains unchanged and equal to $1/\gamma$, where

$$\gamma = \pi^4 EI/(kl^4)$$

Then we find from (1.7) that

$$\gamma\beta^2 < 2\delta\eta_1^{-1}\,(1-\alpha+\varkappa\gamma^{-1}) \tag{2.6}$$
$$\eta_1 = 1 - L - \alpha + \varkappa\gamma^{-1}$$

It is seen that the condition of stability for a viscoelastic rod in the quasistatic formulation of the problem differs from the analogous condition (2.2) found for the dynamic formulation of the same problem.

If we set the constants $\varkappa$ and L in (2.6) equal to zero, i.e. if we consider an elastic rod in a viscous medium, then we get [3, 4]

$$\alpha < 1 - {}^1/_2\gamma\beta^2$$

3. THE STATIONARY REGIME

If the initial deflection of the axis of the rod is non-zero ($f_0 \neq 0$), then the trajectories of variation of the first- and second-order statistical moments of f_1, f_2, z_1 can be determined from the system of equations (1.5), (1.6). If we restrict ourselves to obtaining a solution only for the stationary regime, then, for the systems in question, the derivatives of the moments can be set equal to zero and then the search for the moments of any order can be reduced to the solution of the corresponding system of linear algebraic equations.

Assuming again that the relaxation kernel of the material has the form of a single exponent and omitting the intermediate calculations, we can write down the solutions of the systems of equations (1.5) and (1.6) for the expectation value and the second-order moment of the amplitude of the additional deflection of the rod:

$$\langle f_1\rangle = \frac{\alpha}{\delta} f_0, \quad \langle f_1^2\rangle = \frac{a}{\delta b} f_0^2$$

$$a = 2\alpha\,[(2\varepsilon\alpha + \beta^2\omega^2)\,\eta + \alpha\,(2\varepsilon + \varkappa)L] + \beta^2\omega^2\delta\eta$$

$$b = 2\delta\,[2\varepsilon\,(1-\alpha) + \varkappa L + 2\varepsilon\varkappa\,(2\varepsilon + \varkappa)/\,\omega^2] - \beta^2\omega^2\eta$$

As can be seen, the moments exist if the above-mentioned conditions for the stability of the rod are satisfied.

We shall consider a few characteristic special cases.

Case 1. An elastic rod ($\varkappa = L = 0$):

$$\langle f_1\rangle = \frac{\alpha}{1-\alpha} f_0$$

$$\langle f_1^2\rangle = \frac{4\varepsilon\alpha^2 + \beta^2\omega^2\,(1+\alpha)}{(1-\alpha)\,[4\varepsilon\,(1-\alpha) - \beta^2\omega^2]} f_0^2$$

Case 2. A viscoelastic rod without external damping:

$$\langle f_1\rangle = \frac{\alpha}{1-L-\alpha} f_0$$

$$\langle f_1^2\rangle = \frac{2\alpha^2\varkappa L + \beta^2\omega^2\,(\delta + 2\alpha)\,\eta_0}{\delta\,[\delta\,(2\varkappa L - \beta^2\omega^2) - \beta^2\varkappa^2]} f_0^2$$

Case 3. A weightless rod in a viscous medium:

$$\langle f_1\rangle = \frac{\alpha}{1-L-\alpha} f_0$$

$$\langle f_1^2\rangle = \{2\alpha\,[(\alpha + \gamma\beta^2)\,\eta_1 + \alpha L] + \delta\eta_1\gamma\beta^2\}\left[2\delta\left(1 - \alpha + \frac{\varkappa}{\gamma}\right) - \gamma\beta^2\eta_1\right]^{-1} f_0^2/\delta$$

For an elastic rod ($\varkappa = L = 0$), we get

$$\langle f_1\rangle = \frac{\alpha}{1-\alpha} f_0$$

$$\langle f_1^2\rangle = \frac{2\alpha\,(\alpha + \gamma\beta^2) + (1-\alpha)\,\gamma\beta^2}{(1-\alpha)\,[2\,(1-\alpha) - \gamma\beta^2]}$$

which is identical with the analogous solution in [3].

The results presented indicate that under stationary conditions the second-order moment $\langle f_1^2 \rangle$ can vary even for constant f_0 over an extremely wide range, depending on the relations between the parameters α, β^2, ε, ω^2, $\varkappa$, L.

REFERENCES

1. POTAPOV V. D., Stability of viscoelastic structures subject to stationary random compressive loads. *Izv. Akad. Nauk SSSR, MTT* 3, 153–159, 1984.
2. DIMENTBERG M. F., *Nonlinear Stochastic Problems for Mechanical Oscillations*. Nauka, Moscow, 1980.
3. POTAPOV V. D., *Stability of Viscoelastic Structure Components*. Stroiizdat, Moscow, 1985.
4. POTAPOV V. D., On the stability of rods under stochastic excitation. *Prikl. Mat. Mekh.* **53**, 6, 1006–1013, 1989.
5. TYLIKOWSKI A., Stability and bounds on motion of viscoelastic columns with imperfections and time-dependent forces. IUTAM Creep in Stuctures IV, pp. 150–151. Abstracts of Lectures, Cracow Univ. Technol., Cracow, 1990.
6. DIMENTBERG M. F., *Random Processes in Dynamic Systems with Variable Parameters*. Nauka, Moscow, 1989.
7. BOLOTIN V. V., *Random Oscillations of Elastic Systems*. Nauka, Moscow, 1979.
8. KHAS'MINSKII R. Z., *The Stability of Systems of Differential Equations under Random Perturbations of their Parameters*. Nauka, Moscow, 1969.

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NON-STATIONARY FRICTIONAL HEATING IN SLIDING COMPRESSIBLE ELASTIC BODIES†

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The redistribution of contact pressure due to the influence of the thermal energy generated by the friction between two sliding elastic isotropic bodies is investigated. The plastic strength of the friction pair can be represented as the sum of the force and temperature components of the stress tensor. A method for controlling plastic deformations connected with wear is proposed.

1. WE CONSIDER the problem of contact between two elastic heterogeneous bodies, one of which is a half-space, while the other one is bounded by an axially-symmetric surface of circular shape. The bodies are in contact under the action of a compressive force P and a shear force fP, f being the coefficient of friction. The surface of the half-space is sliding at a constant speed V on the stationary axially symmetric surface (an irregularity) in the direction of the x axis. As a result of friction, heat is generated within the area of contact, which gives rise to the heat flux

$$Q(r) = \gamma f V p(r), \quad r \leqslant a \tag{1.1}$$

into the stationary body. Here γ is distribution coefficient of the heat flux, $p(r)$ is the contact pressure in the corresponding isothermal contact problem [1]

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